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Scaling and statistical geometry in passive scalar turbulence

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We show that the statistics of a turbulent passive scalar at scales larger than the pumping may exhibit multiscaling due to a weaker mechanism than the presence of statistical conservation laws. We develop a general formalism to give explicit predictions for the large scale scaling exponents in the case of the Kraichnan model and discuss their geometric origin at small and large scale.

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Turbulent transport poses challenges for fundamental research with important implications for many environmental (e.g., impact of natural and anthropogenic pollutants on climate) and industrial (e.g., design of effective mixers of chemical products) applications. During the last fifteen years, the field has seen major developments [1]. The study of an analytical tractable model, the Kraichnan model of passive advection [2,3], permitted [4,5] to prove that the statistics of a turbulent passive field (e.g., the temperature) is *intrinsically* not self-similar in the inertial range (fine scales of fluid motion not affected by thermal dissipation). More importantly, drawing on concepts and methods from stochastic analysis [6,7] pointed out a general mechanism accounting for the experimentally and numerically observed multiscaling (see, e.g., [8,9]) of inertial range statistical indicators. Accordingly, the statistics of equal time correlation functions is dominated by global statistical invariants of the Lagrangian dynamics [6,10]. Although this picture can be established in a mathematically controlled way only for the Kraichnan model, numerical investigations of passive scalar advected by the Navier-Stokes equations [11] together with experiments [9,12] give strong evidences of the generality of the mechanism. In the unfolding of these developments, thoroughly summarized in [10], much attention has been devoted to the turbulent inertial range. However, in many physical contexts (e.g., the study of the large scale structures in cosmology [13]) it is important to understand the defining properties of statistical indicators of fluid tracers at scales larger than the typical energy source. As the energy of tracers transported by an incompressible velocity field is expected to "cascade" toward finer scale, one might be tempted to infer from the absence of a "constant-flux" solution of the type predicted by Komogorov's 1941 theory [14] the onset of a thermodynamical equilibrium with Gaussian statistics and equipartition of scalar variance. However it was recently shown analytically [15] and numerically [16] that the presence of an equipartitionlike scalar power spectrum may well coexist with higher-order correlation functions exhibiting breakdown of self-similarity and multiscaling. Underlying these results is the existence, predicted in [6] for the Kraichnan model, of an asymptotic zero-mode expansion of correlation functions *also* at scales larger than the pumping. Here, we device a formalism to calculate (perturbatively) the scaling dimensions of the large scale zero modes. We show that large scale zero modes are not *global* statistical conservation laws of the Lagrangian dynamics. They share however with inertial zero modes a geometrical origin indicated by their being in first approximation specified by eigenvalues of quadratic Casimir's of classical groups. Finally we provide numerical evidence of large scale zero-mode dominance and discuss the relevance of these results for advection by Navier-Stokes. The passive advection of a scalar quantity by a Newtonian incompressible fluid is governed by the equation

$$\partial_t \theta + \boldsymbol{v} \cdot \partial \theta - \frac{\kappa}{2} \partial^2 \theta = f, \tag{1}$$

where $\boldsymbol{v}$ is a vector field solving the Navier-Stokes equation and f is a stochastic large scale stirring. Following Kraichnan [2,3] we model turbulent fluctuations of $\boldsymbol{v}$ by a Gaussian statistics with zero average and

$$\prec v^{\alpha}(\boldsymbol{x},t) v^{\beta}(\boldsymbol{y},s) \succ \; = \delta(t-s) D^{\alpha\beta}_{(\xi)}(\boldsymbol{x}-\boldsymbol{y},m), \tag{2}$$

where the spatial part of the velocity correlation is scale invariant up to an inverse integral scale m^{-1}. Such behavior is encoded in the Mellin representation [17]

$$\tilde{D}^{\alpha\beta}_{(\xi)}(\boldsymbol{x};m,z) \coloneqq \int_0^{\infty} \frac{dw}{w} \frac{D^{\alpha\beta}_{(\xi)}(w\boldsymbol{x};m)}{w^z} = - \frac{D_0 \xi m^{z-\xi} \bar{C}(z,\xi)}{z-\xi} \int \frac{d^d q}{(2\pi)^d} \frac{e^{\imath \boldsymbol{q}\cdot\boldsymbol{x}}}{q^{d+z}} \Pi^{\alpha\beta}(\hat{\boldsymbol{q}}), \tag{3}$$

where $\Pi^{\alpha\beta}$ denotes the Fourier space transversal projector. If $D^{\alpha\beta}_{(\xi)}$ decays faster than power law for $mx \gg 1$ as we suppose here, $\bar{C}(z,\xi)$ is a meromorphic function analytic for Re $z \in (-\infty,0)$ and analytic nonvanishing for $\xi \in [0,2)$. The residues of the simple poles for Re $z=0$ and ξ yield the inertial range asymptotics [17]. For the statistics of the forcing field f we hypothesize time decorrelation (to preserve Galilean invariance), parity, and translational invariance and correlation functions with support peaked around an integral scale $\bar{m}^{-1} \ll m^{-1}$. Mathematically, Eq. (1) is a stochastic partial differential in Stratonovich sense [18] in order to preserve the hydrodynamic interpretation. A straightforward application of the Ito lemma (see, e.g., [10,17]) yields the Hopf equations satisfied by the scalar correlation function $\mathcal{C}_n$ of n fields:

$$\left\{\partial_t - \frac{1}{2}\sum_{i\neq j}^{n} D^{\alpha\beta}_{(\xi)}(\boldsymbol{x}_{ij};m)\partial_{x_i^\alpha}\partial_{x_j^\beta} - \frac{\varkappa^{(\xi)}_{\kappa,m}}{2}\Delta_n\right\}\mathcal{C}_n = \mathfrak{F}_n \tag{4}$$

with Δ_n as the Laplacian in $\mathbb{R}^{nd}$, $\boldsymbol{x}_{ij} := \boldsymbol{x}_i - \boldsymbol{x}_j$, Einstein convention on contracted indices and $\mathfrak{F}_n$ as an effective forcing depending at most on $\mathcal{C}_{n-2}$. The eddy diffusivity $\varkappa^{(\xi)}_{\kappa,m} := \kappa + D^{\alpha}_{(\xi)_\alpha}(0;m)/d$ has a finite inviscid limit $\varkappa^{(\xi)}_{0,m}$ for all $\xi \in [0,2]$. Translational invariance reduces the left-hand side of Eq. (4) to $(\partial_t - M_n^{(\xi)})\mathcal{C}_n$, with $M_n^{(\xi)}$ as a degenerate elliptic operator (for vanishing κ and generic ξ) in $d_n := (n-1)d$ spatial dimensions [6]. The null space of $M_n^{(\xi)}$ can be thought as consisting of local martingales of an effective purely multiplicative stochastic process for each value of n. The relevance of these quantities for the unique solution [19] in $\mathbb{L}^2(\mathbb{R}^{d_n})$ of Eq. (4) is discussed in details in [6,10]. The limit $\xi \downarrow 0$ illustrates the situation. In such a limit [5] $D^{\alpha\beta}_{(0)}$ vanishes for every finite point separation while still contributing to a scale-independent inviscid eddy diffusivity $\varkappa = \varkappa^{(0)}_{0,m}$. Parametrizing $\mathbb{R}^{d_n}$ with Jacobi variables (see, e.g., [20]) $\boldsymbol{R} = (\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{n-1})$, $\boldsymbol{W} = (\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{n-1})$, the reduction in the free Green's function to the translational invariant sector reads as [21]

$$M_n^{(0)^{-1}}(\boldsymbol{R}-\boldsymbol{W}) = \sum_{J=0}^{\infty}\sum_{\boldsymbol{L}} \frac{2\mathcal{K}_{J\boldsymbol{L}}(\boldsymbol{R})\mathcal{H}^{\dagger}_{J\boldsymbol{L}}(\boldsymbol{W})}{\varkappa(d_n+2J-2)} \tag{5}$$

for $R := \|\boldsymbol{R}\| \geq W := \|\boldsymbol{W}\|$. The $\mathcal{H}_{J\boldsymbol{L}}$s are harmonic polynomials providing a complete orthonormal basis of $\mathrm{SO}(d_n)$ through the relation $\mathcal{H}_{J\boldsymbol{L}}(\boldsymbol{R}) = R^J \mathcal{Y}_{J\boldsymbol{L}}(\hat{\boldsymbol{R}})$ (here $\boldsymbol{R} := R\hat{\boldsymbol{R}}$) with hyperspherical harmonics labeled by $d_n - 1$ integers $(J,\boldsymbol{L})$ (see, e.g., [20]). The $\mathcal{K}_{J\boldsymbol{L}}$s are decaying harmonic functions in a one-to-one correspondence with the $\mathcal{H}_{J\boldsymbol{L}}$s specified by the so-called Kelvin transform [22],

$$\mathcal{K}_{J\boldsymbol{L}}(\boldsymbol{R}) = R^{2-d_n}\mathcal{H}_{J\boldsymbol{L}}(\boldsymbol{R}/R^2). \tag{6}$$

The $\mathrm{SO}(d_n)$ decomposition of the Mellin transform of $\mathcal{F}_n$

$$\tilde{\mathcal{F}}_n(\boldsymbol{R},\bar{z}) = \bar{m}^{-\eta_{\mathcal{F}}}\sum_{J\boldsymbol{L}} (\bar{m}R)^{\bar{z}}\mathcal{Y}_{J\boldsymbol{L}}(\hat{\boldsymbol{R}})F_{J\boldsymbol{L}}(\bar{z}) \tag{7}$$

for $\eta_{\mathcal{F}}$ the canonical dimension of $\mathcal{F}_n$ allows us to couch the steady-state solution of Eq. (4) for vanishing ξ as

$$\tilde{\mathcal{C}}_n^{(0)}(\boldsymbol{R},\bar{z}) = \sum_{J\boldsymbol{L}} \frac{2\bar{m}^{-\eta_F}R^2(\bar{m}R)^{\bar{z}}F_{J\boldsymbol{L}}(\bar{z})\mathcal{Y}_{J\boldsymbol{L}}(\hat{\boldsymbol{R}})}{\varkappa(d_n+J+\bar{z})(J-2-\bar{z})}. \tag{8}$$

Equations (7) and (8) can be thought as functionals of identical Lagrangian particles in the unique steady state. Thus there and in the following, for each $J \in \mathbb{N}$ the sum over $\boldsymbol{L}$ is restricted to fully symmetric states. To each hyperangular sector is associated a strip of analyticity, determined by the convergence of the Mellin integral, of size $-d_n - J < \mathrm{Re}\,\bar{z} < J-2$. The simple poles marking the boundary of the strip determine the noncanonical scaling dimensions of the large $\mathcal{K}_{J\boldsymbol{L}}$ and small scale $\mathcal{H}_{J\boldsymbol{L}}$ zero modes. Thus, expansion (8) evinces the geometrical origin, $\mathrm{SO}(d_n)$ anisotropy, of nondimensional scaling. Both classes of zero modes are local martingales as they belong to the null space of Δ_{n-1}. However *only* the $\mathcal{H}_{J\boldsymbol{L}}$ are *strict* martingales, i.e., are preserved by the propagator $P_t := \exp(t\Delta_{n-1})$ of the diffusion: $\mathcal{H}_{J\boldsymbol{L}} = P_t \star \mathcal{H}_{J\boldsymbol{L}}$. A direct calculation shows that projecting first P_t onto its $(J,\boldsymbol{L})$ component renders the convolution $P_t \star \mathcal{K}_{J\boldsymbol{L}}$ integrable but restricts the region where the martingale property is satisfied to a domain $R^2 \gg \varkappa t$ monotonically decreasing in time. The $\mathcal{K}_{J\boldsymbol{L}}$ are therefore *strictly local martingales* [23]. The perturbative construction below in the text suggests that large scale zero modes *are not* expected in general to be statistical conservation laws of the dynamics. At small but finite ξ the $\mathrm{SO}(d_n)$ symmetry is broken to $\sigma_n \times \mathrm{SO}(d)$, with σ_n as the permutation group of n particles. As first shown in [5] solutions of Eq. (4) can be constructed in a systematic perturbation theory in ξ. Combining Eq. (5) with Eq. (7) yields for the $J\boldsymbol{L}$ component of $\mathcal{C}_n = \mathcal{C}_n^{(0)} + \xi\mathcal{C}_n^{(1)} + O(\xi^2)$ in the steady state

$$\begin{aligned}\mathcal{C}^{(1)}_{n,J\boldsymbol{L}}(R,z,\bar{z}) = &-\frac{\mathcal{C}^{(0)}_{n,J\boldsymbol{L}}(R,\bar{z})\ln m}{z} \\ &-\frac{2^{z/2}n(n-1)R^2(mR)^z(\bar{m}R)^{\bar{z}}C(z)}{z^2(d_n+J+z+\bar{z})(J-2-z-\bar{z})} \\ &\times\sum_{a=1}^{2}\int d\Omega_{d_n}\mathcal{Y}^{\dagger}_{J\boldsymbol{L}}(\hat{\boldsymbol{W}})\mathbf{J}_{aa}\mathcal{D}_a\mathcal{C}^{(0)}_n(\boldsymbol{W},\bar{z})\Big|^{W=1}_{\bar{m}=1},\end{aligned} \tag{9}$$

with $\mathcal{D}_a := w_1^z\{\delta^{\alpha\beta} - z/(d-1+z)w_1^\alpha w_1^\beta/w_1^2\}\partial_{w_a^\alpha}\partial_{w_a^\beta}$ and $C(z)$ such that $C(0)=1$. In deriving Eq. (9) we adopted an orthonormal set of Jacobi variables such that $\boldsymbol{r}_1 := \boldsymbol{x}_{12}$ and $\boldsymbol{r}_2 := [(n-2)(\boldsymbol{x}_1+\boldsymbol{x}_2) - 2\Sigma_{j=3}^n \boldsymbol{x}_j]/\sqrt{2(n-2)n}$. In such a case the Jacobian of the change in variables give only two nonvanishing contributions $(\mathbf{J}_{11},\mathbf{J}_{22})$ equal to $(\frac{1}{2},\frac{n-2}{2n})$. The order of evaluation of the residues in the Mellin variables $z,\bar{z}$ determines the order of the limits of vanishing m and $\bar{m}$. The condition $m \ll \bar{m}$ is enforced evaluating first the residue for z equal zero. Corrections to scaling are then associated to *double poles* in $\bar{z}$ occurring only for $\bar{z}_{J,+} = J-2$ (inertial range) and $\bar{z}_{J,-} = -d_n - J$ (large scales). Thus it is sufficient to diagonalize Eq. (9) in the $\mathrm{SO}(d_n)$ representation specified by J. Universal terms in the two asymptotics, labeled by $i = \{+,-\}$, are encoded into finite dimensional matrices I_i depending upon the asymptotics and the $\mathrm{SO}(d_n)$ representation:

$$\begin{aligned}\mathcal{C}^{(1)}_{n;J\boldsymbol{L}}(\boldsymbol{R},i) \to \frac{2\bar{m}^{-\eta_{\mathcal{F}}}R^{2+\bar{z}_{J,i}}}{\varkappa(d_n+2J-2)}\Bigg\{&F_{J\boldsymbol{L}}(\bar{z}_{J,i})\ln\frac{\bar{m}}{\sqrt{2}} \\ &-(-1)^i\ln(\bar{m}R)\sum_{\boldsymbol{L}'}\langle J,\boldsymbol{L}|I_i|J,\boldsymbol{L}'\rangle F_{J\boldsymbol{L}'}(\bar{z}_{J,i})\Bigg\} + \cdots.\end{aligned} \tag{10}$$

The "$\cdots$" stand for nonlogarithmic corrections. Scaling exponents are determined by the eigenvalues $\zeta^{(1)}_{\bar{z}_{J,i}}$ of I_i according to $\zeta_{\bar{z}_{J,i}} = 2 + \bar{z}_{J,i} + \xi\zeta^{(1)}_{\bar{z}_{J,i}} + O(\xi^2)$. It is expedient to choose a representation of hyperspherical harmonics adapted to the group-subgroup chain adapted to $\mathrm{SO}(d_n) \supset \mathrm{SO}(d)^{n-1}$ (see, e.g., [20,21]). If we focus on the SO(d)-isotropic sector of $\mathcal{C}_4$ as in [5] for permutation invariant states the representation is two dimensional and all calculations can be performed explicitly [24]. The inertial range asymptotics recovers the results

$$\zeta^{(1)}_{4,+}([4,0]) = -\frac{2(d+4)}{d+2}, \quad \zeta^{(1)}_{4,+}([4,2]) = -\frac{2(d-2)}{d-1}, \tag{11}$$

respectively, corresponding to the irreducible and reducible zero modes [5]. The large scale asymptotics yields

$$\zeta^{(1)}_{4,-}([4,0]) = \frac{d+6}{d+2}, \quad \zeta^{(1)}_{4,-}([4,2]) = \frac{d-3}{d-1}. \tag{12}$$

In order to interpret the results and justify the notation, we rewrite the scalar products on the d_n hypersphere in Eq. (9) in terms of the Gaussian measure of $\mathbb{R}^{d_n}$ so that for any $\varepsilon > 0$

$$\langle J,\boldsymbol{L}|I_{-}|J,\boldsymbol{L}'\rangle = \sum_{a=1}^{2}\frac{2n(n-1)\mathbf{J}_{aa}}{d_n+2J-2}\,\frac{d}{dz}\Bigg|_{\substack{z=0\\ \bar z=-d_n-J}} \times \int d^{d_n}W\frac{e^{-W^2/2R_o^2}W^{\varepsilon}\mathcal{H}^{\dagger}_{J\boldsymbol{L}}(\boldsymbol{W})\mathcal{D}_a W^{2+\bar z}\mathcal{Y}_{J\boldsymbol{L}'}(\hat{\boldsymbol{W}})}{(2R_o^2)^{(z+\bar z+J+\varepsilon)/2}\Gamma\left(\frac{z+\bar z+J+\varepsilon}{2}\right)} \tag{13}$$

so that we can integrate by parts in the Cartesian coordinates. By incompressibility of Eq. (2) the operation reduces to letting $\mathcal{D}_a$ act to its left in Eq. (13). Projecting back to the SO(d_n) scalar product and taking the limit of vanishing ε yield the relation $\langle J,\boldsymbol{L}|I_{-}|J,\boldsymbol{L}'\rangle = \langle J,\boldsymbol{L}'|I_{+}-1|J,\boldsymbol{L}\rangle$ implying $\zeta^{(1)}_{\bar z_{J,-}} = -\zeta^{(1)}_{\bar z_{J,+}} - 1$ satisfied by Eqs. (11) and (12) so that $\zeta_{\bar z_{J,-}} + \zeta_{\bar z_{J,+}} = 2 - d_n - \xi + O(\xi^2)$ which is consistent with the nonperturbative analysis of [6]. In the literature (see, e.g., [25,26]) the $\zeta_{\bar z_{J,+}}$s have been computed in general for irreducible zero modes [5,10] as they are the only to contribute to structure functions. Here we outline a different approach based on the martingale property of the $\mathcal{H}_{J\boldsymbol{L}}$s and conceptually "dual" to the Wilsonian renormalization of composite operators of [17]. Instead of studying operators of the renormalized theory with larger infrared cutoff, we study martingales of the original theory in the limit of infinite integral scale. To this goal we introduce the infrared regularized harmonic polynomials $\mathcal{H}^{[L]}_{J\boldsymbol{L}}(\boldsymbol{R}) \coloneqq \mathcal{H}_{J\boldsymbol{L}}(\boldsymbol{R})\exp\{-R^2/(2L^2)\}$. These are eigenstates of the isotropic harmonic oscillator in $\mathbb{R}^{d_n}$ and, consequently, eigenstates of the Fourier transform [27]. Using this property and the diagrammatic techniques of [17] it is straightforward to evaluate the convolutions

$$\lim_{L\uparrow\infty} M_n^{(0)^{-1}} \star \frac{\mathcal{H}^{[L]}_{J\boldsymbol{L}}}{L^2} = \frac{2\mathcal{H}_{J\boldsymbol{L}}}{\varkappa(d_n+2J-2)} \tag{14}$$

and for $J>0$

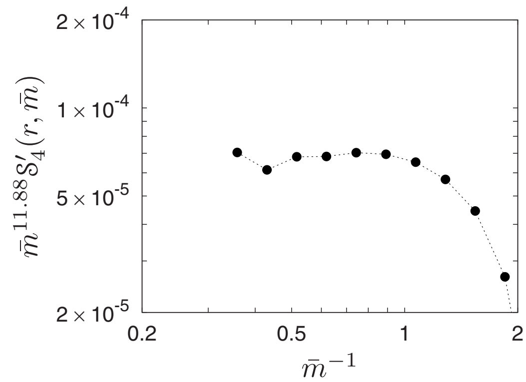

FIG. 1. Numerical large scale behavior of $\mathcal{S}'_4(r,\bar m) \coloneqq \mathcal{S}_4(r,\bar m) - \mathcal{C}_4(0,\bar m)$ (four-point structure function minus four-point correlation at coinciding points) versus the integral scale $\bar m^{-1}$ balanced by the theoretical zero-mode prediction $\bar m^{\zeta_{4,-}([4,0])}$, with $\zeta_{4,-}([4,0]) = 2-\xi-\zeta_{4,-}([4,0])$. The plot is obtained by averaging over $N=10^9$ Lagrangian paths using the algorithm of [29,30] at $\xi=0.4$, $r=1$, and $d=3$. By Eq. (12) $\zeta_{4,-}([4,0]) = 11.88 + O(\xi^2)$. Forcing is non-Gaussian and proportional to the hyperspherical harmonic $\mathcal{Y}_{4,\boldsymbol{L}^\star}$ specifying the zeroth order of the *irreducible* inertial range zero mode (see [5] for details). The observed behavior significantly deviates from the scaling prediction coming from the exponent $2-d_4-\xi$ of the Green's function.

$$\lim_{L\uparrow\infty} M_n^{(1)^{-1}}(z) \star \frac{\mathcal{H}^{[L]}_{J\boldsymbol{L}}}{L^2} = -\frac{2\mathcal{H}_{J\boldsymbol{L}}\ln m}{z\varkappa(d_n+2J-2)} - \sum_{l\neq k}\sum_{a,b}^{n-1}\mathbf{J}^{(lk)}_{a1}\mathbf{J}^{(lk)}_{b1}\frac{\partial_{r^{\alpha}_{a;(lk)}}\partial_{r_{b;(lk)}\beta}\mathcal{H}_{J\boldsymbol{L}}(\boldsymbol{R}_{(lk)})}{z\varkappa(d_n+2J-2)} \times \frac{d\bar C(z,0)m^z}{(d-1)}\int\frac{d^dq}{(2\pi)^d}\frac{2^{2+(z/2)}e^{\imath\boldsymbol{q}\cdot\boldsymbol{r}_{1;(lk)}}\Pi^{\alpha\beta}(\hat{\boldsymbol{q}})}{\bar C(0,0)q^{d+z+2}}, \tag{15}$$

where $\mathbf{J}^{(lk)}$ is the Jacobian of orthonormal Jacobi coordinates adapted to $\boldsymbol{r}_{1;(lk)} = \boldsymbol{x}_{lk}/\sqrt{2}$. The integral in Eq. (15) yields the first term of the loop expansion to which the perturbative theory for the $\mathcal{C}_n$s reduces if the limit $\bar m \downarrow 0$ is taken first. The integral may seem to require analyticity of $\bar C(z,0)$ in the strip Re $z\in[-2,0)$. However the residue for Re $z=-2$ is proportional to $\Delta_{n-1}\mathcal{H}_{J\boldsymbol{L}}$ and vanishes. The scaling dimensions of the inertial range zero modes are determined by prefactor of the self-similarity breaking term $\ln m$. After some algebra we get into

$$\operatorname{Res}_{z=0}\left\{\lim_{L\uparrow\infty} M_n^{(1)^{-1}}(z) \star \frac{\mathcal{H}^{[L]}_{J\boldsymbol{L}}}{L^2}\right\} = \frac{2\ln m}{\varkappa(d_n+2J-2)}\left\{1 + \frac{(d+1)\mathfrak{C}^{(2,n)}_{\mathrm{SO}(d)} - d\,\mathfrak{C}^{(2,n)}_{\mathrm{SU}(d)}}{2(d-1)(d+2)}\right\}\mathcal{H}_{J\boldsymbol{L}} + \cdots, \tag{16}$$

with $\mathfrak{C}^{(2,n)}_{\mathrm{SU}(d)}=\mathfrak{C}^{(2,n)}_{\mathrm{SU}(n-1)}+\frac{(d+1-n)}{d(n-1)}\mathfrak{E}(\mathfrak{E}+d_n)$, $\mathfrak{E}$ as the generator of dilations and $\mathfrak{C}^{(2,n)}_{\mathrm{SO}(d)}$ and $\mathfrak{C}^{(2,n)}_{\mathrm{SU}(n-1)}$ as the mutually commuting quadratic Casimir invariants of SO(d) and SU($n-1$) acting on translation and permutation invariant homogeneous polynomials of n particle variables in d dimensions. Although $[\Delta_n,\mathfrak{C}^{(2,n)}_{\mathrm{SU}(n-1)}]\neq 0$, any homogeneous polynomial P of degree J admits a unique expansion $P_J=\sum_{k=0}^{k_\star}R^{2k}H_{J-2k}$, $k_\star=\mathrm{int}(J/2)$ for the H_Js harmonic homogeneous polynomials of degree J [22]. Thus linear combinations of the $\mathcal{H}_{JL}$s specify eigenstates of the Casimir invariants up to *slow modes* of the free theory [6]. By Gel'fand-Zetlin theory (see, e.g., [28]) the eigenvalues are $\lambda_{\mathrm{SO}(d)}(j)=j(j+d-2)$ and $\lambda_{\mathrm{SU}(n-1)}(\boldsymbol{a})=\sum_{i=1}^{n-2}a_i(a_i-2i)+\frac{J[(n-1)n-J]}{n-1}$ for j, $\boldsymbol{a}=[a_1,\ldots,a_{n-2}]$ nonnegative integers satisfying $\sum_{i=1}^{n-2}a_i=J$ and $a_i\geq a_j$ for any $i\geq j$ so that

$$\zeta^{(1)}_{\bar{z}_{J,+}}(j,\boldsymbol{a})=\frac{(d+1)j(j+d-2)}{2(d-1)(d+2)}-\frac{d\sum_{i=1}^{n-2}a_i(a_i-2i)+J[d(d+1)-J]}{2(d-1)(d+2)}. \quad (17)$$

Irreducible zero modes correspond to $\boldsymbol{a}=[n,0,\ldots,0]$ ($J=n$ and $n-3$ zeroes), while the four point reducible zero mode correspond to $\boldsymbol{a}=[2,2]$. For $\mathcal{C}_2$ [15,16] the value of the forcing spectrum at zero momentum determines whether the decay at scales larger than the pumping is power law or exponential, in the latter case paving the way for anisotropic scaling dominance. Figure 1 illustrates realizability of large scale anomalous scaling for $\mathcal{C}_4$ and non-Gaussian forcing.

These results give an analytical though perturbative validation of the general link between geometry and intermittency in passive scalar turbulence numerically established in [11]. Furthermore, in the inertial range the above analysis carries over to a passive scalar advected by the Navier-Stokes equation in the thermal stirring regime forced by a Gaussian random field self-similar with Hölder exponent ε. As shown in [31], at leading order in a loop expansion in ε the scalar is driven only by the Gaussian core of the velocity statistics described by a Kraichnan model with $\xi\propto\varepsilon$.

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